

Exotic sheaves via categorical actions

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- ▶ But hard to understand (for a geometer).

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Question

Do exotic sheaves come from categorical actions?

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Study representations V by decomposing into weight spaces V_λ and analyzing the action of \mathfrak{sl}_2 -triples.

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Categorify:

$V_\lambda \rightsquigarrow$ (triangulated) categories $\mathcal{K}(\lambda)$

$e_i, f_i \rightsquigarrow$ adjoint functors E_i, F_i

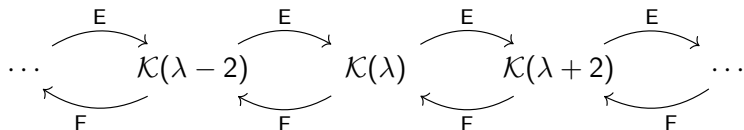
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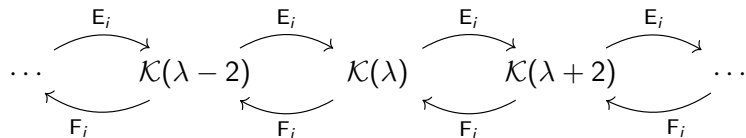
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$$EF|_{\mathcal{K}(\lambda)} \cong FE|_{\mathcal{K}(\lambda)} \oplus \bigoplus_{\lambda} \text{Id}_{\mathcal{K}(\lambda)}$$

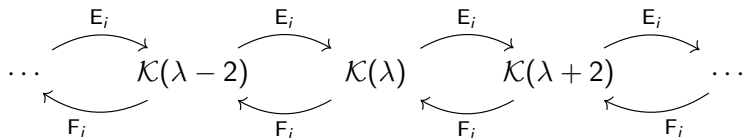
Braid groups



Form complexes of functors $\mathcal{K}(\lambda) \rightarrow \mathcal{K}(s_i \lambda)$

$$T_i|_{\mathcal{K}(\lambda)} = E_i^{(\ell)} \rightarrow F_i E_i^{(\ell+1)} \rightarrow F_i^{(2)} E_i^{(\ell+2)} \rightarrow \dots, \quad \ell = \langle \alpha_i, \lambda \rangle.$$

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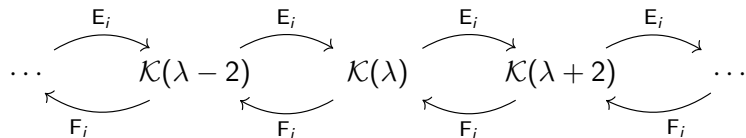
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Theorem (Cautis, Kamnitzer)

The T_i induce an action of the braid group of \mathfrak{g} on $\bigoplus_{\lambda} \mathcal{K}(\lambda)$.

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Corollary

$\widehat{\mathfrak{sl}}_n$ -action \rightsquigarrow affine braid group action

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abelian subcat. of $\mathcal{K}(\mu)$ $\xrightarrow[\text{to exact functors}]{E_i, F_i \text{ restrict}}$ abelian subcat. of each $\mathcal{K}(\lambda)$

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In particular: Exotic sheaves can be obtained in this way.

Applications

- ▶ Geometric construction of categories of exotic sheaves.
- ▶ Study these categories inductively starting from the easy highest weight categories.
- ▶ Get exotic sheaves on new spaces (convolution varieties of affine Grassmannian orbit closures).
- ▶ Applications to knot theory?
- ▶ Applications to birational geometry?