Exotic sheaves via categorical actions

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Exotic sheaves

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- But hard to understand (for a geometer).

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Question

Do exotic sheaves come from categorical actions?

Categorical actions

Usual representation theory of a Lie algebra $\mathfrak{g}$

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Categorify:

$$V_\lambda \rightsquigarrow \text{(triangulated) categories } \mathcal{K}(\lambda)$$

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$$\cdots \underset{\mathsf{F}}{\overset{\mathsf{E}}{\rightleftarrows}} \mathcal{K}(\lambda-2) \underset{\mathsf{F}}{\overset{\mathsf{E}}{\rightleftarrows}} \mathcal{K}(\lambda) \underset{\mathsf{F}}{\overset{\mathsf{E}}{\rightleftarrows}} \mathcal{K}(\lambda+2) \underset{\mathsf{F}}{\overset{\mathsf{E}}{\rightleftarrows}} \cdots$$

$$\mathsf{EF}|_{\mathcal{K}(\lambda)} \cong \mathsf{FE}|_{\mathcal{K}(\lambda)} \oplus \bigoplus_{\lambda} \mathsf{Id}_{\mathcal{K}(\lambda)}$$

Braid groups

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Form complexes of functors $\mathcal{K}(\lambda) \to \mathcal{K}(s_i\lambda)$

$$\mathsf{T}_i|_{\mathcal{K}(\lambda)} = \mathsf{E}_i^{(\ell)} \to \mathsf{F}_i\mathsf{E}_i^{(\ell+1)} \to \mathsf{F}_i^{(2)}\mathsf{E}_i^{(\ell+2)} \to \cdots, \quad \ell = \langle \alpha_i, \lambda \rangle.$$

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Corollary

$\widehat{\mathfrak{sl}}_n$*-action* $\rightsquigarrow$ *affine braid group action*

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Fix μ such that $\mathcal{K}(\mu)$ is "big enough":

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In particular: Exotic sheaves can be obtained in this way.

Applications

- Geometric construction of categories of exotic sheaves.
- Study these categories inductively starting from the easy highest weight categories.
- Get exotic sheaves on new spaces (convolution varieties of affine Grassmannian orbit closures).
- Applications to knot theory?
- Applications to birational geometry?