D-MODULES

NOTES

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OCTOBER 22, 2020

CONTENTS

1. INTRODUCTION

1.1. SOME REFERENCES

We will mainly follow the book [HTT]. Other references include [A; B1; B2; BGKHME; BCEY; G; MS; M]. Please be aware that some of these documents – while undoubtedly very useful – do occasionally contain errors.

1.2. WARNING

These are work-in-progress lecture notes and hence will contain an above average amount of errors. Please send any corrections to clemens@koppensteiner.site.

2. THE RING OF DIFFERENTIAL OPERATORS

Unless otherwise mentioned, throughout this course X will be a quasi-projective smooth complex variety. Alternatively, many – but not all – of the results also hold for smooth analytic varieties. We will write Θ_X for the tangent bundle of X and Ω^1_X for its cotangent bundle. Both are locally free of rank $\dim X$. The tangent bundle Θ_X has a natural action on the ring of regular functions $\mathcal{O}_X$ by differentiation.

Definition 2.1. The *ring of differential operators* $\mathscr{D}_X$ is subalgebra of $\mathscr{E}nd_{\mathbb{C}}(\mathscr{O}_X)$ generated by $\mathscr{O}_X$ and Θ_X.

Thus, if one picks local coordinates $x_1, \dots, x_n$ of X, the sheaf $\mathscr{D}_X$ is locally the free algebra generated by $x_1, \dots, x_n$ and $\partial_i = \frac{\partial}{\partial x_i}$, $1 \leq i \leq n$, subject to the relations

$$
\begin{aligned}
[x_i, x_j] &= 0 && \text{for all } i, j, \\
[\partial_i, \partial_j] &= 0 && \text{for all } i, j, \\
[\partial_i, x_i] &= 1 && \text{for all } i, \\
[\partial_i, x_j] &= 0 && \text{for all } i \neq j.
\end{aligned}
$$

The sheaf of $\mathbb{C}$-algebras $\mathscr{D}_X$ has a natural filtration $F_i\mathscr{D}_X = \mathscr{D}_X^{\leq i}$ by degree of the differential operator. In other words $F_0\mathscr{D}_X = \mathscr{O}_X$ and

$$F_{i+1}\mathscr{D}_X = \{\theta \in \mathscr{D}_X : [\theta, f] \in F_i\mathscr{D}_X \text{ for all } f \in \mathscr{O}_X\}.$$

The associated graded $\operatorname{gr}\mathscr{D}_X$ is naturally identified with

$$\pi_*\mathscr{O}_{T^*X} \cong \operatorname{Sym}_{\mathscr{O}_X}\Theta_X.$$

Corollary 2.2. *$\mathscr{D}_X$ is Noetherian of weak global dimension at most* $2\dim X$.

We can now introduce our main objects of study.

Definition 2.3. The abelian category of (left) $\mathscr{D}_X$-modules (or *D-modules* for short) is denoted by $\mathbf{Mod}(\mathscr{D}_X)$.

Examples 2.4. Clearly $\mathscr{D}_X$ is a left module over itself. The structure sheaf $\mathscr{O}_X \cong \mathscr{D}_X/F_{\geq 1}\mathscr{D}$ is a D-module with the usual action of differential operators on functions.

Given any closed subvariety Z defined by a sheaf of ideals $\mathscr{I}_Z$, we can consider the module $\mathscr{D}_X/\mathscr{D}_X\mathscr{I}_Z$. In particular, if $X = \mathbb{A}^1$ and $Z = \{0\}$ is the origin, $\mathscr{M} = \mathscr{D}_X/\mathscr{D}_X\mathscr{I}_Z$ is given by $\mathbb{C}[\partial]$, where $x \in \mathscr{O}_X$ acts on $\mathscr{M}$ by $x \cdot \partial^n = -n\partial^{n-1}$. This is the skyscraper D-module at the origin (one notes that the usual skyscraper $\mathscr{O}$-module $\mathbb{C}_0$ cannot be made into a D-module). ○

Example 2.5. Consider a differential operator $P \in \mathscr{D}_X$ and the left D-module $\mathscr{M} = \mathscr{D}_X/\mathscr{D}_X P$. Then we have

$$
\begin{aligned}
\operatorname{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X) &= \operatorname{Hom}(\mathscr{D}_X/P\mathscr{D}_X, \mathscr{O}_X) \\
&\cong \{\varphi \in \operatorname{Hom}_{\mathscr{D}_X}(\mathscr{D}_X, \mathscr{O}_X) : \varphi(P) = 0\} \\
&\cong \{f \in \mathscr{O}_X : Pf = 0\},
\end{aligned}
$$

where the last isomorphism comes from the identification $\operatorname{Hom}_{\mathscr{D}_X}(\mathscr{D}_X, \mathscr{O}_X) \cong \mathscr{O}_X$, $\varphi \mapsto \phi(1)$: $Pf = P\varphi(1) = \varphi(P1) = \varphi(P) = 0$. Therefore D-modules know about solutions to differential equations. ○

Given any integrable connection $\mathscr{F} \to \mathscr{F} \otimes \Omega^1_X$, one dually obtains a map $\nabla\colon \Theta_X \to \mathscr{E}nd_{\mathbb{C}}(\mathscr{F})$. One checks that this can be upgraded to an action of $\mathscr{D}_X$ on $\mathscr{M}$. In fact, we can alternatively describe D-modules as integrable connections whose underlying $\mathscr{O}_X$-modules are not required to be locally free.

Lemma 2.6. *Giving a left $\mathscr{D}_X$-module structure on an $\mathscr{O}_X$-module $\mathscr{M}$ is equivalent to giving a $\mathbb{C}$-linear morphism*

$$\nabla\colon \Theta_X \to \mathscr{E}nd_{\mathbb{C}}(\mathscr{M}), \quad \theta \to \nabla_\theta$$

such that the following conditions hold for all $\theta \in \Theta_X$, $f \in \mathscr{O}_X$ and $s \in \mathscr{M}$:

(i) $\nabla_{f\theta}(s) = f\nabla_\theta(s)$,

(ii) $\nabla_\theta(fs) = \theta(f)s + f\nabla_\theta(s)$,

(iii) $\nabla_{[\theta_1,\theta_2]}(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s)$.

While one often prefers to work with left D-modules, we will see that many constructions are more natural to define for right D-modules. We will write $\mathbf{Mod}(\mathscr{D}_X^{\mathrm{op}})$ for the category or right $\mathscr{D}_X$-modules.

Exercise 2.7. Give a description of right D-modules analogous to Lemma 2.6.

Example 2.8. We write $\omega_X = \bigwedge^{\dim X} \Omega^1_X$ for the canonical line bundle on X. There exists a natural action of Θ_X on ω_X via the *Lie derivative*:

$$\mathrm{Lie}_\theta(\omega)(\theta_1, \dots, \theta_n) = \theta(\omega(\theta_1, \dots, \theta_n)) - \sum_{i=1}^{n} \omega(\theta_1, \dots, [\theta, \theta_i], \dots, \theta_n),$$

for $\omega \in \omega_X$, $\theta_i \in \Theta_X$ and $n = \dim X$. This gives ω_X the structure of a right $\mathscr{D}_X$-module by

$$\omega\theta := -\mathrm{Lie}_\theta(\omega).$$

○

Lemma 2.9. *Let $\mathscr{M}$, $\mathscr{M}' \in \mathbf{Mod}(\mathscr{D}_X)$ and $\mathscr{N}$, $\mathscr{N}' \in \mathbf{Mod}(\mathscr{D}_X^{\mathrm{op}})$. Then with $\theta \in \Theta_X$ one has the following module structures:*

$$\begin{aligned}
\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{M}' &\in \mathbf{Mod}(\mathscr{D}_X), & (m \otimes m')\theta &:= \theta m \otimes m' + m \otimes \theta m', \\
\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{M} &\in \mathbf{Mod}(\mathscr{D}_X^{\mathrm{op}}), & (n \otimes m)\theta &:= n\theta \otimes m - n \otimes \theta m, \\
\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{M}') &\in \mathbf{Mod}(\mathscr{D}_X), & (\theta\phi)(m) &:= \theta(\phi(m)) - \phi(\theta m), \\
\mathscr{H}om_{\mathscr{O}_X}(\mathscr{N}, \mathscr{N}') &\in \mathbf{Mod}(\mathscr{D}_X), & (\theta\phi)(n) &:= -\phi(n)\theta + \phi(n\theta), \\
\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N}) &\in \mathbf{Mod}(\mathscr{D}_X^{\mathrm{op}}), & (\phi\theta)(m) &:= \phi(m)\theta + \phi(\theta m).
\end{aligned}$$

Exercise 2.10. Show that there exists a canonical isomorphism

$$\mathscr{D}^{\mathrm{op}} \cong \omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X \otimes_{\mathscr{O}_X} \omega_X^\vee.$$

Lemma 2.11. *Let* $\mathcal{M}, \mathcal{M}' \in \mathbf{Mod}(\mathcal{D}_X)$ *and* $\mathcal{N} \in \mathbf{Mod}(\mathcal{D}_X^{\mathrm{op}})$*. Then there exist isomorphisms*

$$(\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{M}') \otimes_{\mathcal{D}_X} \mathcal{M} \cong \mathcal{N} \otimes_{\mathcal{D}_X} (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M}') \cong (\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{M}) \otimes_{\mathcal{D}_X} \mathcal{M}'.$$

Lemma 2.12. *The assignment*

$$\mathcal{M} \mapsto \omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

extends to an equivalence of categories $\mathbf{Mod}(\mathcal{D}_X) \to \mathbf{Mod}(\mathcal{D}_X^{\mathrm{op}})$ *with quasi-inverse given by*

$$\mathcal{N} \mapsto \omega_X^{\vee} \otimes_{\mathcal{O}_X} \mathcal{N} := \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{N}).$$

We will frequently make use of this equivalence to freely switch between the perspectives of right and left modules. This will be of particular importance in the next sections when we define various functors acting on D-modules.

Definition 2.13. The full subcategory of $\mathbf{Mod}(\mathcal{D}_X)$ consisting $\mathcal{O}_X$-quasi-coherent D-modules is denoted by $\mathbf{Mod}_{\mathrm{qc}}(\mathcal{D}_X)$.

A $\mathcal{D}_X$-module $\mathcal{M}$ is *coherent* if its is locally finitely presented and for any open subset U of X any locally finitely generated submodule of $\mathcal{M}|_U$ is locally finitely presented. We write $\mathbf{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ for the corresponding full subcategory of $\mathbf{Mod}(\mathcal{D}_X)$.

Fact 2.14. *A* $\mathcal{D}_X$*-module is coherent if and only if it is* $\mathcal{O}_X$*-quasi-coherent and locally finitely generated over* $\mathcal{D}_X$*. In particular* $\mathcal{D}_X$ *is coherent as a module over itself.*

Fact 2.15. *Any* $\mathcal{O}_X$*-coherent* $\mathcal{D}_X$*-module is locally free as an* $\mathcal{O}_X$*-module (and finitely generated).*

Definition 2.16. We write $\mathrm{D}(\mathcal{D}_X)$ for the derived category of $\mathbf{Mod}(\mathcal{D}_X)$. We denote by $\mathrm{D}_{\mathrm{qc}}(\mathcal{D}_X)$ and $\mathrm{D}_{\mathrm{coh}}(\mathcal{D}_X)$ the full subcategories of $\mathrm{D}(\mathcal{D}_X)$ consisting of those complexes with cohomology sheaves in $\mathbf{Mod}_{\mathrm{qc}}(\mathcal{D}_X)$ and $\mathbf{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ respectively. As usually we add a superscript $+$, $-$, or b to indicated boundedness conditions.

Fact 2.17. *The natural functors*

$$\mathrm{D^b}(\mathbf{Mod}_{\mathrm{qc}}(\mathcal{D}_X)) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_X)$$

$$\mathrm{D^b}(\mathbf{Mod}_{\mathrm{coh}}(\mathcal{D}_X)) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$$

are equivalences.

3. PUSHFORWARD AND PULLBACK

Given a morphism $f\colon X \to Y$ of smooth complex varieties, we want to define functors between the corresponding (derived) categories of D-modules. It turns out that that the most natural functors to define actually go between the categories of *right* D-modules. It is however common to apply the side-switching operations and work with left modules instead, so we will also do so here.

3.1. THE TRANSFER MODULES

Let $f\colon X \to Y$ be a morphism of smooth complex varieties and let $\mathscr{M}$ be a left $\mathscr{D}_Y$-module. We note that there is a canonical $\mathscr{O}_X$-module morphism $f^*\Omega^1_Y \to \Omega^1_X$. Taking its dual, we obtain a canonical morphism

$$f' : \Theta_X \to f^*\Theta_Y.$$

Using this we can endow the $\mathscr{O}_X$-module

$$f^*\mathscr{M} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{M}$$

with the structure of a left $\mathscr{D}_X$-module: given $\theta \in \Theta_X$ and $s\otimes m \in f^*\mathscr{M} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{M}$ we set

$$\theta(s \otimes m) = \theta(s) \otimes m + sf'(\theta)(m),$$

where for $f'(\theta) = \sum t_i \otimes \theta_i$ we set $sf'(\theta)(m) = \sum st_i \otimes \theta_i(m)$.

If $\mathscr{M}$ is also a right $\mathscr{D}_Y$-module, then $f^*\mathscr{M}$ continues to be a right $f^{-1}\mathscr{D}_Y$-module by the right action on $f^{-1}\mathscr{M}$. In particular we obtain a $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodule $f^*\mathscr{D}_Y$.

Applying the side-switching operations, we also obtain an $(f^{-1}\mathscr{D}_Y, \mathscr{D}_X)$-bimodule

$$\omega_X \otimes_{\mathscr{O}_X} f^*\mathscr{D}_Y \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\omega_Y^\vee.$$

Definition 3.1. The bimodules

$$\mathscr{D}_{X\to Y} := f^*\mathscr{D}_Y$$

and

$$\mathscr{D}_{Y\leftarrow X} := \omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_{X\to Y} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\omega_Y^\vee$$

are called the *transfer bimodules.*

Exercise 3.2. Consider the closed embedding $i\colon \mathbb{A}^{n-k} \hookrightarrow \mathbb{A}^n$ as the $x_1 = \dots = x_k = 0$. Show that

$$\mathscr{D}_{\mathbb{A}^{n-k}\to\mathbb{A}^n} \cong \mathscr{D}_{\mathbb{A}^{n-k}} \otimes_{\mathbb{C}} \mathbb{C}[\partial_1, \dots, \partial_k]$$

as a left $\mathscr{D}_{\mathbb{A}^{n-k}}$-module.

3.2. PULLBACK

Definition 3.3. Define the *pullback* (or *inverse image*) functor

$$f^! : \mathrm{D}^-(\mathscr{D}_Y) \to \mathrm{D}^-(\mathscr{D}_X), \qquad f^!\mathscr{M} = \mathscr{D}_{X\to Y} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{M}[\dim X - \dim Y].$$

Remark 3.4. Why the shift and why the notation $f^!$? Grothendieck duality gives a functor $f^!_{\mathscr{O}}$ between categories of $\mathscr{O}$-modules. It turns out that this functor is compatible with *right* D-module structures. Thus we get a functor between left modules by applying the side-switching operations:

$$\omega_X^\vee \overset{\mathbb{L}}{\otimes}_{\mathscr{O}_X} f^!_{\mathscr{O}} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathscr{O}_Y} f^{-1}\omega_Y.$$

However, since everything is smooth, we have isomorphisms

$$f^!_{\mathscr{O}}(-) \cong f^!_{\mathscr{O}}\mathscr{O}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{O}_X} f^*(-) \cong \omega_{X/Y} \otimes f^*(-)[\dim X - \dim Y]$$

and

$$\omega_{X/Y} \cong \omega_X \otimes_{\mathscr{O}_X} f^*\omega_Y^\vee \cong \omega_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\omega_Y^\vee.$$

Thus the canonical bundles cancel out and we are left with the definition above.

Ultimately this choice will be justified by compatibility with the de Rham functor.

Remark 3.5. In [HTT], the functor $f^!$ is denoted $f^\dagger$. In [B1], the "naive pullback" f^* is denoted f^Δ, while in some other places it is also denoted $f^\dagger$.

Lemma 3.6. *The functor $f^!$ restricts to a functor $\mathrm{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathscr{D}_Y) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathscr{D}_X)$.*

Remark 3.7. It does not however restrict to a functor $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. For example, if $f\colon X \to Y$ is a non-trivial closed immersion, the pullback of $\mathscr{D}_Y$ is not $\mathscr{D}_X$-coherent: according to Exercise 3.2 it is a locally free $\mathscr{D}_X$-module of infinite rank.

Exercise 3.8. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of smooth complex varieties. Show that $f^! \circ g^! \cong (g \circ f)^!$.

Proposition 3.9. *Let $i\colon Z \hookrightarrow X$ be a closed embedding of smooth varieties and $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathscr{D}_X)$. Then we have a canonical isomorphism*

$$i^!\mathscr{M} \cong \mathbb{R}\mathscr{H}om_{i^{-1}\mathscr{D}_X}(\mathscr{D}_{X\leftarrow Z}, i^{-1}\mathscr{M}).$$

Proof. To simplify notation, we will show this for the corresponding functors of right modules. Applying tensor-Hom adjunction one has

$$\begin{aligned}
\mathbb{R}\mathscr{H}om_{i^{-1}\mathscr{D}_X}(\mathscr{D}_{Z\to X}, i^{-1}\mathscr{M}) &= \mathbb{R}\mathscr{H}om_{i^{-1}\mathscr{D}_X}(\mathscr{O}_X \otimes_{i^{-1}\mathscr{O}_X} i^{-1}\mathscr{D}_X, i^{-1}\mathscr{M}) \\
&\cong \mathbb{R}\mathscr{H}om_{i^{-1}\mathscr{O}_X}(\mathscr{O}_X, \mathbb{R}\mathscr{H}om_{i^{-1}\mathscr{D}_X}(i^{-1}\mathscr{D}_X, i^{-1}\mathscr{M})) \\
&\cong \mathbb{R}\mathscr{H}om_{i^{-1}\mathscr{O}_X}(\mathscr{O}_X, i^{-1}\mathscr{M}) \\
&\cong i^!_{\mathscr{O}}\mathscr{M}.
\end{aligned}$$

To complete the proof, one needs to check that this identification is compatible with the D-module structures. Instead of doing so, we will give an explicit proof of the result via a Koszul-type resolution of $\mathscr{D}_{X\leftarrow Z}$ in the exercises. ☐

Consider a product $X \times Y$ and let p_1, p_2 be the two projection maps. For $\mathscr{M} \in \mathbf{Mod}(\mathscr{D}_X)$, $\mathscr{N} \in \mathbf{Mod}(\mathscr{D}_Y)$ define

$$\mathscr{M} \boxtimes \mathscr{N} := \mathscr{D}_{X\times Y} \underset{p_1^{-1}\mathscr{D}_X \underset{\mathbb{C}}{\otimes} p_2^{-1}\mathscr{D}_Y}{\otimes} (p_1^{-1}\mathscr{M} \underset{\mathbb{C}}{\otimes} p_2^{-1}\mathscr{N}).$$

The underlying $\mathscr{O}_{X\times Y}$ of $\mathscr{M} \boxtimes \mathscr{N}$ is the same as the $\mathscr{O}$-module box product of $\mathscr{M}$ and $\mathscr{N}$. As $\boxtimes$ is exact in both arguments, it immediately extends to the derived categories.

Lemma 3.10. *Let $\mathscr{M}, \mathscr{N} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$.*

(i) Denote by $\Delta\colon X \to X \times X$ the diagonal morphism. Then $\mathscr{M} \overset{\mathbb{L}}{\otimes}_{\mathscr{O}_X} \mathscr{N} \cong \mathbb{L}\Delta^*(\mathscr{M} \boxtimes \mathscr{N})$.

(ii) Let $f\colon Y \to X$ be a morphism. Then $\mathbb{L}f^*(\mathscr{M} \overset{\mathbb{L}}{\otimes}_{\mathscr{O}_X} \mathscr{N}) \cong \mathbb{L}f^*\mathscr{M} \overset{\mathbb{L}}{\otimes}_{\mathscr{O}_Y} \mathbb{L}f^*\mathscr{N}$.

3.3. PUSHFORWARD

Consider a right $\mathscr{D}_X$-module $\mathscr{N}$. Recall that the transfer module $\mathscr{D}_{X\to Y}$ is a $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$ bimodule. Thus $\mathscr{N} \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y}$ is a right $f^{-1}\mathscr{D}_Y$-module. We can therefore define a pushforward functor for *right* D-modules

$$\mathscr{N} \mapsto f_*(\mathscr{N} \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y}) \in \mathbf{Mod}(\mathscr{D}_Y^{\mathrm{op}}).$$

As one commonly works with *left* D-modules, we use the transfer modules to obtain the corresponding functor for left D-modules

$$\mathscr{M} \mapsto \omega_Y^\vee \otimes_{\mathscr{O}_Y} f_*((\omega_X \otimes_{\mathscr{O}_X} \mathscr{M}) \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y}).$$

To make this a bit less unwieldy, we use Lemma 2.11 and the projection formula [H, Exercise II.5.1] to rewrite the result as

$$\begin{aligned}\omega_Y^\vee \otimes_{\mathscr{O}_Y} f_*((\omega_X \otimes_{\mathscr{O}_X} \mathscr{M}) \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y}) &\cong \omega_Y^\vee \otimes_{\mathscr{O}_Y} f_*((\omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_{X\to Y}) \otimes_{\mathscr{D}_X} \mathscr{M})\\ &\cong f_*(f^{-1}\omega_Y^\vee \otimes_{f^{-1}\mathscr{O}_X} \omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_{X\to Y}) \otimes_{\mathscr{D}_X} \mathscr{M})\\ &\cong f_*(\mathscr{D}_{Y\leftarrow X} \otimes_{\mathscr{D}_X} \mathscr{M}).\end{aligned}$$

Definition 3.11. Define the *pushforward* (or *direct image*) functor

$$f_\bullet \colon \mathrm{D^b}(\mathscr{D}_X) \to \mathrm{D^b}(\mathscr{D}_Y), \quad f_\bullet(\mathscr{M}) = \mathbb{R}f_*(\mathscr{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}).$$

We note that the definition of $f_\bullet$ contains a left-derived functor and a right derived functor, which can make it a bit tricky to handle.

Remark 3.12. In [HTT] and many other text the functor $f_\bullet$ is denoted by $\int_f$. One should be careful not to confuse $f_\bullet$ with the functor f_* for plain sheaves (or $\mathscr{O}$-modules). In particular, if $\pi : X \to \mathrm{pt}$ is the structure map, then $\pi_\bullet\mathscr{M}$ does not compute derived global sections. Rather, as we will see later, it will compute de Rham cohomology.

Exercise 3.13. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of smooth complex varieties. Show that $g_\bullet \circ f_\bullet \cong (g \circ f)_\bullet$.

Example 3.14. If $j\colon U \hookrightarrow X$ is an open immersion, then $j^*\mathscr{D}_X = j^{-1}\mathscr{D}_X = \mathscr{D}_U$. Hence $j_\bullet = \mathbb{R}j_*$. ○

Example 3.15. If $i\colon Z \hookrightarrow X$ is a closed immersion, then Exercise 3.2 shows that locally

$$H^0(i_\bullet\mathscr{M}) \cong \mathbb{C}[\partial_1, \dots, \partial_k] \otimes_{\mathbb{C}} i_*\mathscr{M}$$

and

$$H^\ell(i_\bullet\mathscr{M}) = 0 \qquad \text{for } \ell \neq 0.$$

○

Corollary 3.16. *Let* $i\colon Z \hookrightarrow X$ *be a closed immersion.*

(i) *For any* $\mathscr{M} \in \mathbf{Mod}(\mathscr{D}_X)$ *we have* $H^\ell(i_\bullet \mathscr{M})$ *for* $\ell \neq 0$. *In particular*

$$H^0(i_\bullet(-)) \colon \mathbf{Mod}(\mathscr{D}_Z) \to \mathbf{Mod}(\mathscr{D}_X)$$

is an exact functor.

(ii) $i_\bullet$ *restricts to a functor* $\mathrm{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathscr{D}_Z) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathscr{D}_X)$.

The following is a consequence of Proposition 3.9.

Proposition 3.17. *Let* $i \colon Z \hookrightarrow X$ *be a closed immersion. There exists a canonical isomorphism*

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(i_\bullet \mathscr{M}, \mathscr{N}) \cong \mathbb{R}i_* \mathbb{R}\mathscr{H}om(\mathscr{M}, i^! \mathscr{N}).$$

In particular $i^!$ *is right adjoint to* $i_\bullet$.

Remark 3.18. Recall that for $\mathscr{O}$-module functors, f^* is *left* adjoint to f_*, while if f is proper, $f^!_{\mathscr{O}}$ is *right* adjoint to $f_\bullet$. Hence the notation $f^!$ for the D-module pullback is apt. We will later see that $f^!$ is right adjoint to $f_\bullet$ for any proper morphism f, at least on the coherent subcategories.

Next, we would like to compute $f_\bullet$ for a projection $f \colon X \times Y \to Y$. To do so, we will start with the following lemma, which is useful in many situations.

Lemma 3.19 (Spencer resolution)**.** *Set* $n = \dim X$. *The complex*

$$0 \to \mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^n \Theta_X \to \ldots \to \mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^0 \Theta_X \to \mathscr{O}_X \to 0$$

is a locally free resolution of the left $\mathscr{D}_X$*-module* $\mathscr{O}_X$. *The complex*

$$0 \to \bigwedge^0 \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \ldots \to \bigwedge^n \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \omega_X \to 0$$

is a locally free resolution of the right $\mathscr{D}_X$*-module* ω_X.

Let us describe the differentials:

$$\mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^0 \cong \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{O}_X \cong \mathscr{D}_X \to \mathscr{O}_X$$

is given by $P \mapsto P(1)$, and

$$d \colon \mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^k \Theta_X \to \mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^{k-1} \Theta_X$$

is given by

$$d(P \otimes \theta_1 \wedge \cdots \wedge \theta_k) =$$
$$\sum_{i=0}^k (-1)^{i+1} P\theta_i \otimes \theta_1 \wedge \cdots \wedge \widehat{\theta_i} \wedge \cdots \wedge \theta_k + \sum_{i<j} (-1)^{i+j} P \otimes [\theta_i, \theta_j] \wedge \theta_i \wedge \cdots \wedge \widehat{\theta_i} \wedge \cdots \wedge \widehat{\theta_j} \wedge \cdots \wedge \theta_k.$$

The map

$$\bigwedge^n \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \cong \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \omega_X$$

is given by $\omega \otimes P \mapsto \omega P$ and the differential

$$d\colon \bigwedge^k \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \bigwedge^{k+1} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

is given in local coordinates $\{z_i, \partial_i\}$ by

$$d(\omega \otimes P) = d\omega \otimes P + \sum_i dz_i \wedge \omega \otimes \partial_i P.$$

Proof of Lemma 3.19. The two complexes differ by the side-changing operations, so it suffices to prove that the first one is acyclic. Let C be that complex and consider the following filtration:

$$F_p C = \left[F_{p-n}\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^n \Theta_X \to \ldots \to F_p\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^0 \Theta_X \to F_p\mathcal{O}_X \right].$$

It now suffices to show that the associated graded is acyclic (spectral sequence of a filtered complex). Let $\pi\colon T^*X \to X$ be the projection and $i\colon X \hookrightarrow T^*X$ the zero section. Then $\operatorname{gr} C \cong \pi_* D$ with

$$D = \left[\mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathcal{O}_X} \bigwedge^n \pi^{-1}\Theta_X \to \ldots \to \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathcal{O}_X} \bigwedge^0 \pi^{-1}\Theta_X \to i_*\mathcal{O}_X \right].$$

But D is just the Koszul resolution of the $\mathcal{O}_{T^*X}$-module $i_*\mathcal{O}_X$, and hence is acyclic. Since π_* is affine, $\operatorname{gr} C$ and C are also acyclic. □

Consider now a projection $f\colon X \times Y \to Y$ and let $\mathcal{M} \in \mathbf{Mod}(\mathcal{D}_{X\times Y})$. We note that $\mathcal{D}_{Y\leftarrow X\times Y} = \omega_X \boxtimes \mathcal{D}_Y$. The Spencer resolution induces a locally free (and hence flat) resolution of $\mathcal{D}_{Y\leftarrow X\times Y} \otimes_{\mathcal{D}_{X\times Y}} \mathcal{M}$. Set $\Omega^k_f = \Omega^k_X \boxtimes \mathcal{O}_Y$.

Definition 3.20. The *relative de Rham complex* of $\mathcal{M}$ is given by

$$\mathrm{DR}_f(\mathcal{M})^k = \begin{cases} \Omega_f^{k+\dim X} \otimes_{\mathcal{O}_{X\times Y}} \mathcal{M} & -\dim X \leq k \leq 0 \\ 0 & \text{otherwise,} \end{cases}$$

with differential

$$d(\omega \otimes m) = d\omega \otimes m + \sum_{i=1}^{\dim X} (dz_i \wedge \omega) \otimes \partial_i s$$

in local coordinates $\{z_i, \partial_i\}$ of X.

Corollary 3.21. *Let $f\colon X \times Y \to Y$ be the projection.*

(i) For $\mathcal{M} \in \mathbf{Mod}(\mathcal{D}_{X\times Y})$ we have $f_\bullet \mathcal{M} \cong \mathbb{R}f_*(\mathrm{DR}_f(\mathcal{M}))$.

(ii) The functor $f_\bullet$ restricts to a functor $\mathrm{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_{X\times Y}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_Y)$.

Remark 3.22. In particular, if $f\colon X \to \mathrm{pt}$ is the structure map, then $f_\bullet \mathscr{O}_X$ computes the (algebraic) de Rham cohomology of X.

Corollary 3.23. *Let $f\colon X \to Y$ be a morphism of smooth varieties. Then $f_\bullet$ restricts to a functor $\mathrm{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathscr{D}_{X\times Y}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathscr{D}_Y)$.*

Proof. We can always factor a morphism into a closed immersion (the graph of f) followed by a projection. We already know the statement for each of these cases. □

3.4. Kashiwara's equivalence

Consider a closed embedding $i\colon Z \hookrightarrow X$. Denote by $\mathbf{Mod}_{\mathrm{qc}}(\mathscr{D}_X)_Z$ the full subcategory of $\mathbf{Mod}(\mathscr{D}_X)$ consisting of D-modules which are (set-theoretically) supported on Z. Similarly, denote by $\mathbf{Mod}_{\mathrm{coh}}(\mathscr{D}_X)_Z$ the corresponding subcategory of $\mathbf{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$.

Theorem 3.24 (Kashiwara's Equivalence)**.** *Let $i\colon Z \hookrightarrow X$ be a closed immersion.*

(i) The functor $H^0 i_\bullet$ induces an equivalence of categories

$$\mathbf{Mod}_{\mathrm{qc}}(\mathscr{D}_X) \xrightarrow{\sim} \mathbf{Mod}_{\mathrm{qc}}(\mathscr{D}_X)_Z,$$

with quasi-inverse given by $H^0 i^!$.

(ii) This equivalence restricts to an equivalence

$$\mathbf{Mod}_{\mathrm{coh}}(\mathscr{D}_X) \xrightarrow{\sim} \mathbf{Mod}_{\mathrm{coh}}(\mathscr{D}_X)_Z.$$

(iii) For any $\mathscr{N} \in \mathbf{Mod}_{\mathrm{qc}}(\mathscr{D}_X)_Z$ one has $H^\ell i^! \mathscr{N} = 0$ for $\ell \neq 0$.

Proof. We already know that $H\ell i_\bullet = 0$ for $\ell \neq 0$. We will first show (i) and (iii). By adjunction we have canonical maps

$$\mathrm{Id} \to H^0 i^! \circ H^0 i_\bullet \qquad \text{and} \qquad H^0 i_\bullet \circ H^0 i^! \to \mathrm{Id}.$$

To show (i), we have to show that these are isomorphisms on the categories in question. This is a local statement. As (iii) is also local, we may shrink X as necessary. Further, by induction on the codimension of Z in X, we may assume that Z is a hypersurface.

We can thus pick local coordinates $\{x_k, \partial_k\}$ on X such that Z is given by $\{x_1 = 0\}$. Set $z = x_1$ and $\partial = \partial_1$.

For $\mathscr{M} \in \mathbf{Mod}(\mathscr{D}_Z)$ and $\mathscr{N} \in \mathbf{Mod}(\mathscr{D}_X)_Z$ we have

$$\begin{aligned} H^0 i_\bullet \mathscr{M} &= \mathbb{C}[\partial] \otimes_{\mathbb{C}} i_* \mathscr{M} \\ H^0 i^! \mathscr{N} &= \ker(z\colon i^{-1}\mathscr{N} \to i^{-1}\mathscr{N}) \\ H^1 i^! \mathscr{N} &= \operatorname{coker}(z\colon i^{-1}\mathscr{N} \to i^{-1}\mathscr{N}) \\ H^\ell i^! \mathscr{N} &= 0 \qquad \text{for } \ell \neq 0, 1, \end{aligned}$$

where the last three assertions follow by taking a free resolution of $\mathscr{D}_{Z\to X} = \mathscr{D}_X / \mathscr{D}_X z$.

Consider now the *Euler operator* $\theta = z\partial$ and form the eigenspaces

$$\mathcal{N}^j = \{s \in \mathcal{N} : \theta s = js\}$$

By $[\partial, z] = 1$, we have $z\mathcal{N}^j \subseteq \mathcal{N}^{j+1}$ and $\partial\mathcal{N}^j \subseteq \mathcal{N}^{j-1}$. Clearly θ acts by multiplication by j on $\mathcal{N}^j$ and thus is an isomorphism for $j \neq 0$. Thus $\partial z = \theta + 1 \colon \mathcal{N}^j \to \mathcal{N}^j$ is an isomorphism for $j \neq -1$. It follows that for $j < -1$ we have isomorphisms

$$z \colon \mathcal{N}^j \xrightarrow{\sim} \mathcal{N}^{j+1} \qquad \text{and} \qquad \partial \colon \mathcal{N}^{j+1} \xrightarrow{\sim} \mathcal{N}^j.$$

We now claim that

$$\mathcal{N} = \bigoplus_{j=1}^{\infty} \mathcal{N}^{-j}. \tag{1}$$

By assumption, $\mathcal{N}$ is quasi-coherent as an $\mathcal{O}_X$-module supported on Z, and hence every section s of $\mathcal{N}$ is annihilated by z^k for some sufficiently large k. Hence it suffices to show that

$$\ker(z^k \colon \mathcal{N} \to \mathcal{N}) \subseteq \bigoplus_{j=1}^{k} \mathcal{N}^{-j}$$

for all $k \geq 1$. We will induct on k. For $k = 1$ the condition $zs = 0$ implies that $\theta s = (\partial z - 1)s = -s$ and hence $s \in \mathcal{N}^{-1}$.

Assume that $k \geq 2$ and let s be a section of $\ker(z^k \colon \mathcal{N} \to \mathcal{N})$. Then $0 = z^k s = z^{k-1} zs$ and by induction $zs \in \bigoplus_{j=1}^{k-1} \mathcal{N}^{-j}$. It follows that

$$\theta s + s = z\partial s_s = \partial zs \in \bigoplus_{j=2}^{k} \mathcal{N}^{-j}.$$

We also have $z^{k-1}(\theta s + ks) = z^k \partial s + kz^{k-1}s = \partial z^k s = 0$. Again by induction we obtain

$$\theta s + ks \in \bigoplus_{j=1}^{k-1} \mathcal{N}^{-j}.$$

Taking the difference between these observations, we see that $(k-1)s \in \bigoplus_{j=1}^{k} \mathcal{N}^{-j}$, and, since $k \geq 2$, the same is true for s. We have thus shown (1).

Since $z \colon \mathcal{N}^j \to \mathcal{N}^{j+1}$ is an isomorphism for $j \leq -2$, (1) immediately implies that

$$H^0 i^! = i^{-1}\mathcal{N}^{-1} \qquad\qquad H^1 i^! = 0,$$

showing (iii). Since also $\partial \colon \mathcal{N}^{j+1} \to \mathcal{N}^j$ is an isomorphism for $j \leq -2$, (1) implies that

$$\mathcal{N} \cong \mathbb{C}[\partial] \otimes_{\mathbb{C}} \mathcal{N}^{-1}.$$

Statement (i) follows.

It remains to show (ii). This is again a local problem. Locally, we have $H^0 i_\bullet \mathcal{M} = \mathbb{C}[\partial] \otimes_{\mathbb{C}} i_* \mathcal{M}$, which is clearly finitely generated as a $\mathcal{D}_X$-module. Conversely, let $\mathcal{N} \in \mathbf{Mod}_{\mathrm{coh}}(\mathcal{D}_X)_Z$. Writing $\mathcal{N} = \mathbb{C}[\partial] \otimes_{\mathbb{C}} \mathcal{N}^{-1}$, we see that there are finitely many generators $s_1, \dots, s_r \in \mathcal{N}^{-1}$ generating $\mathcal{N}$. Their images in $H^0 i^! \mathcal{N} = i^{-1}\mathcal{N}^{-1}$ also generate. $\square$

Remark 3.25. We saw in the proof that the Euler operator $z\partial$ only has negative integer eigenvalues on $\mathscr{N}$. The filtration $V_\bullet$ of $\mathscr{N}$ defined by $V_k\mathscr{N} = \sum_{j=-\infty}^{k} \mathscr{N}^j$ is called the *Kashiwara–Malgrange V-filtration*. It exists more generally for $\mathscr{D}_X$-modules which are not necessarily supported on Z, but the eigenvalues of $z\partial$ will usually not be integers. Nevertheless some of the basic properties we showed hold in more generality, and play an important role when understanding D-modules on compactifications (and hence in Hodge theory).

As usual, Theorem 3.24 has a derived counterpart, which one obtains by induction on the cohomological length of a complex. For this let $\mathrm{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathscr{D}_X)_Z$ be the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathscr{D}_X)$ consisting of complexes $\mathscr{M}$ whose cohomology modules $H^\ell(\mathscr{M})$ are contained in $\mathbf{Mod}_{\mathrm{qc}}(\mathscr{D}_X)_Z$. Define $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)_Z$ analogously.

Corollary 3.26. *Let $i\colon Z \hookrightarrow X$ be a closed immersion. Then the functor*

$$i_\bullet\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathscr{D}_Z) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathscr{D}_X)_Z$$

is a equivalence of triangulated categories with quasi-inverse given by $i^!$. This equivalence restricts to an equivalence

$$i_\bullet\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Z) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)_Z.$$

A. DERIVED CATEGORIES

Let $\mathbf{A}$ be an abelian category. Homological algebra tells us that we should look at the category of complexes $\mathbf{Kom}(\mathbf{A})$, but of course this category isn't quite the right thing to look at. For example we want to identify homotopic morphisms. But before we do that let us introduce some additional structure.

Firstly, we have a shift endofunctor: $A^\bullet[1] = A^{\bullet+1}$ with differential multiplied by (-1).

Second, given a morphism of complexes $f\colon A^\bullet \to B^\bullet$ we can form the cone

$$\mathrm{cone}(f) = A^\bullet[1] \oplus B^\bullet$$

with differential given by $\begin{pmatrix} d_A[1] & 0 \\ f & d_B \end{pmatrix}$:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{-d_A^{n-1}} & A^n & \xrightarrow{-d_A^{n}} & A^{n+1} & \xrightarrow{-d_A^{n+1}} & A^{n+2} & \xrightarrow{-d_A^{n+2}} & \cdots \\
 & \searrow^{f^{n-1}} & \oplus & \searrow^{f^{n}} & \oplus & \searrow^{f^{n+1}} & \oplus & \searrow^{f^{n+2}} & \\
\cdots & \xrightarrow{d_B^{n-2}} & B^{n-1} & \xrightarrow{d_B^{n-1}} & B^n & \xrightarrow{d_B^{n}} & B^{n+1} & \xrightarrow{d_B^{n+1}} & \cdots
\end{array}$$

We obtain a triangle of morphisms

$$\begin{array}{ccc}
 & A & \\
{}^{+1}\nearrow & & \searrow^{f} \\
\mathrm{cone}(f) & \longleftarrow & B
\end{array}$$

Recall that two morphisms of complexes $f, g \colon A \to B$ are homotopic if there exits a collection of maps $h^\bullet \colon A^\bullet \to B^{\bullet-1}$ such that $f - g = d_B h + h d_A$. In this case we write $f \sim g$.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_A^{n-2}} & A^{n-1} & \xrightarrow{d_A^{n-1}} & A^n & \xrightarrow{d_A^{n}} & A^{n+1} & \xrightarrow{d_A^{n+1}} & \cdots \\
\swarrow & h^{n-1} & f^{n-1} \downarrow\downarrow g^{n-1} & h^{n} \swarrow & f^{n} \downarrow\downarrow g^{n} & h^{n+1} \swarrow & f^{n+1} \downarrow\downarrow g^{n+1} & h^{n+2} \swarrow & \\
\cdots & \xrightarrow{d_B^{n-2}} & B^{n-1} & \xrightarrow{d_B^{n-1}} & B^n & \xrightarrow{d_B^{n}} & B^{n+1} & \xrightarrow{d_B^{n+1}} & \cdots
\end{array}$$

We write $\mathbf{K}(\mathbf{A})$ for the homotopy category of complexes, i.e. the category with the same objects as $\mathbf{Kom}(\mathbf{A})$ and with morphisms

$$\mathrm{Hom}_{\mathbf{K}(\mathbf{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathbf{Kom}(\mathbf{A})}(A^\bullet, B^\bullet)\big/\sim .$$

We call any triangle $X \to Y \to Z \to X[1]$ which is isomorphic to a triangle $X \xrightarrow{f} Y \to \mathrm{cone}(f) \to X[1]$ in $\mathbf{K}(\mathbf{A})$ a *distinguished triangle*. The shift functor and the collection distinguished triangles give $\mathbf{K}(\mathbf{A})$ the structure of a *triangulated category*. Thus it is an additive category and satisfies the following axioms.

(TR$_1$) For any object X, the triangle $X \xrightarrow{\mathrm{Id}} X \to 0$ is distinguished. For any morphism $f \colon X \to Y$ there exists a distinguished triangle $X \xrightarrow{f} Y \to Z$ (Z is called a *mapping cone of f*). Any triangle that is isomorphic to a distinguished triangle is distinguished.

(TR$_2$) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is distinguished, then so are the rotated triangles

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1] \quad \text{and} \quad Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z.$$

(TR$_3$) Given two triangles and maps f and g which make the left-most square in the diagram below commute, there exists a (not necessarily unique) morphism h making everything commute

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
\end{array}$$

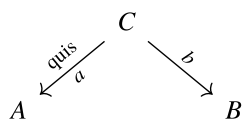
(TR$_4$) The octahedral axiom. Given maps $f \colon X \to Y$ and $g \colon Y \to Z$ the mapping cones of f, g and gf fit into a distinguished triangle $\mathrm{cone}(f) \to \mathrm{cone}(gf) \to \mathrm{cone}(g)$, so that all possible diagrams commute. (These diagrams can be drawn in the shape of an octagon.)

Note that it follows from (TR$_3$) that any two mapping cones of a morphism f are isomorphic, but not necessarily uniquely so. In particular, a morphism f is an isomorphism if and only if $\mathrm{cone}(f) = 0$. We also note that $\mathbf{Kom}(\mathbf{A})$ does not satisfy (TR$_1$), as the cone of the identity morphism is only null-homotopic, but not isomorphic to zero.

A functor between triangulated categories $F\colon \mathbf{S} \to \mathbf{T}$ together with an isomorphism $\phi\colon F \circ [1] \cong [1] \circ F$ is called *exact* (or *triangulated*) if it is additive and sends distinguished triangles to distinguished triangles.

A morphism of complexes is called a *quasi-isomorphism* if it induces an isomorphism on cohomology. We want to identify quasi-isomorphic objects and thus formally invert all quasi-isomorphisms.

Definition A.1. The derived category $\mathrm{D}(\mathbf{A})$ of $\mathbf{A}$ has the same objects as $\mathbf{K}(\mathbf{A})$ and morphisms $X \to Y$ in $\mathrm{D}(\mathbf{A})$ are roofs



where a is a homotopy class of a quasi-isomorphism $C \to A$ and b is a homotopy class of a morphism $C \to B$.

We write $\mathrm{D}^+(\mathbf{A})$, $\mathrm{D}^-(\mathbf{A})$ and $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$ for the full subcategories of $\mathrm{D}(\mathbf{A})$ consisting of objects $X \in \mathrm{D}(\mathbf{A})$ such that $H^i(X) = 0$ for all $i \ll 0$, resp. all $i \gg 0$, resp. all i with $|i| \gg 0$.

There exists a natural quotient functor $\mathbf{K}(\mathbf{A}) \to \mathrm{D}(\mathbf{A})$. We declare any triangle which is isomorphic to the image of a distinguished triangle under this morphism (i.e., to a triangle of the form $X \xrightarrow{f} Y \to \mathrm{cone} f)$ to be a distinguished triangle in $\mathrm{D}(\mathbf{A})$. This gives $\mathrm{D}(\mathbf{A})$ the structure of a triangulated category. The same is true for the various bounded versions, which are full triangulated subcategories of $\mathrm{D}(\mathbf{A})$.

If $F\colon \mathbf{A} \to \mathbf{B}$ is a left exact functor we define its *right derived functor* $RF\colon \mathrm{D}^+(\mathbf{A}) \to \mathrm{D}^+(\mathbf{B})$ by $RF(X) = F(I)$, where I is a complex consisting of injective objects with a quasi-isomorphism $X \xrightarrow{\sim} I$ and $F(I)$ is to be understood component-wise. Similarly, we define the *left derived functor* $LF\colon \mathrm{D}^-(\mathbf{A}) \to \mathrm{D}^-(\mathbf{B})$ of a right exact functor F as $F(P)$ for a projective resolution $P \xrightarrow{\sim} X$.

In particular, we obtain a bifunctor $R\operatorname{Hom}(-,-)$.

Exercise A.2. Let $A = k[\epsilon]/(\epsilon^2)$ be the dual numbers. Show that in $\mathrm{D}^{\mathrm{b}}(\mathbf{Mod}(A))$ the complexes $0 \to k \xrightarrow{0} k \to 0$ and $0 \to A \xrightarrow{\epsilon} A \to 0$ are not quasi-isomorphic, despite having isomorphic cohomology modules. [Hint: Compute the endomorphism rings of the second complex. Note that this is easy to do since it is a bounded complex of projective A-modules. Deduce that it is indecomposable.]

The above assumes that the category has enough injectives and projectives respectively. For categories of sheaves this is not always the case. Thus one uses F-injective (resp. F-projective) complexes (see for example [w, Theorem 10.5.9]. For example, we can compute the derived tensor product $\otimes^L$ with a flat (e.g. locally free) resolution.

Warning A.3. The above only applies to the appropriately bounded derived category. Unbounded derived categories can behave in unexpected ways. For example the complex

$$\cdots \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \cdots$$

is an acyclic complex, i.e. it is quasi-isomorphic to the zero complex in $D(\mathbf{Mod}(\mathbb{Z}/4))$. But while the complex consists of free modules, it cannot be used to compute the derived tensor product: tensoring the complex with $\mathbb{Z}/2$ gives

$$\cdots \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \cdots,$$

which is not acyclic.

REFERENCES

[A] Sergey Arkhipov. *Topics in Algebraic Geometry: $\mathscr{D}$-modules*. Lecture Notes.

[B1] J. Bernstein. *Algebraic Theory of D-modules*. URL: http://math.uchicago.edu/~drinfeld/langlands/Bernstein/Bernstein-dmod.ps.

[B2] J.-E. Björk. *Rings of differential operators*. North-Holland Mathematical Library 21. North-Holland Publishing Co., Amsterdam-New York, 1979, pp. xvii+374. ISBN: 0-444-85292-1.

[BCEY] Alexander Braverman, Tatyana Chmutova, Pavel Etingof, and David Yan. *Introduction to algebraic D-modules*. URL: https://math.mit.edu/~xxu/Course/18.748/EtingofLec.pdf.

[BGKHME] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, and F. Ehlers. *Algebraic D-modules*. Perspectives in Mathematics 2. Academic Press, Inc., Boston, MA, 1987. ISBN: 0-12-117740-8.

[G] Victor Ginzburg. *Lectures on $\mathscr{D}$-modules*. Lecture notes. URL: http://www.math.harvard.edu/~gaitsgde/grad_2009/Ginzburg.pdf.

[H] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics 52. New York: Springer-Verlag, 1977. xvi+496. ISBN: 0-387-90244-9.

[HTT] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. *D-modules, perverse sheaves, and representation theory*. Progress in Mathematics 236. Translated from the 1995 Japanese edition by Takeuchi. Boston, MA: Birkhäuser Boston Inc., 2008. xii+407. ISBN: 978-0-8176-4363-8. DOI: 10.1007/978-0-8176-4523-6.

[M] Zoghman Mebkhout. *Le formalisme des six opérations de Grothendieck pour les $\mathscr{D}_X$-modules cohérents*. Travaux en Cours [Works in Progress] 35. With supplementary material by the author and L. Narváez Macarro. Hermann, Paris, 1989. ISBN: 2-7056-6049-6.

[MS] Philippe Maisonobe and Claude Sabbah. *Aspects of the theory of $\mathscr{D}$-modules. Lecture notes (Kaiserslautern 2002)*. July 2011. URL: http://www.cmls.polytechnique.fr/perso/sabbah.claude/livres/kaiserslautern.pdf.

[w] Charles A. Weibel. *An introduction to homological algebra*. Cambridge Studies in Advanced Mathematics 38. Cambridge University Press, Cambridge, 1994. ISBN: 0-521-43500-5; 0-521-55987-1. DOI: 10.1017/CBO9781139644136.