D-MODULES: EXERCISE SHEET 2

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Exercise 1. A *rigid dualizing complex* over $\mathscr{D}_X$ is a dualizing complex $\mathscr{R}$ together with a rigidifying isomorphism

$$\rho\colon \mathscr{R} \xrightarrow{\sim} \mathbb{R}\mathscr{H}\!om_{\mathscr{D}_X \otimes \mathscr{D}_X^{\mathrm{op}}}(\mathscr{D}_X, \mathscr{R} \otimes \mathscr{R}).$$

(We remark that rigid dualizing complexes are unique up to unique isomorphism.) Show that $\mathscr{D}_X[2\dim X]$ is rigid. [Hint: Let $\Delta\colon X \to X \times X$ be the diagonal morphism and use that $\mathbb{D}_{X\times X}\Delta_{\bullet}\mathscr{O}_X \cong \Delta_{\bullet}\mathbb{D}_X\mathscr{O}_X \cong \Delta_{\bullet}\mathscr{O}_X$.]

Exercise 2. Let $\mathscr{M}$ be an integrable connection, i.e. an $\mathscr{O}_X$-coherent $\mathscr{D}_X$-module. Show that

$$\mathbb{D}\mathscr{M} \cong \mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X).$$

Exercise 3. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module with $\mathrm{Ch}(\mathscr{M}) \subseteq X = T^*_X X$. Show that $\mathscr{M}$ is an integrable connection.

Exercise 4. Let $i\colon Z \hookrightarrow X$ be a closed embedding. Let

$$T^*Z \xleftarrow{\ \rho\ } i^*(T^*X) = Z \otimes_X T^*X \xrightarrow{\ \varpi\ } T^*X$$

be the natural morphisms induced by i. Let $\mathscr{M}$ be a coherent $\mathscr{D}_Z$-module. Show that

$$\mathrm{Ch}(i_{\bullet}\mathscr{M}) = \varpi(\rho^{-1}\,\mathrm{Ch}(\mathscr{M})).$$

Deduce that $\mathscr{M}$ is holonomic if and only if $i_{\bullet}\mathscr{M}$ is.

Exercise 5. Show that for any affine open subset U of X the ring $\mathscr{D}_X(U)$ has left and right global dimension $\dim X$. Deduce that any $\mathscr{M} \in \mathbf{Mod}_{\mathrm{qc}}(\mathscr{D}_X)$ has a locally projective resolution of length at most $\dim X$.

Exercise 6. Show that there exists a *canonical* morphism of functors

$$f_! \to f_{\bullet}\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{\mathscr{D}_Y}).$$