First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your **simplified** answers in the boxes.
   Marking scheme: 1 for each correct, 0 otherwise

   (a) Compute the derivative of $\left(\dfrac{7x+2}{x^2+3}\right)$

   Answer: $\frac{21-4x-7x^2}{(x^2+3)^2}$

   Solution: We use quotient rule:

   $$\frac{(x^2+3)\cdot 7 - 2x\cdot(7x+2)}{(x^2+3)^2} = \frac{21-4x-7x^2}{(x^2+3)^2}$$

   (b) Evaluate $\lim_{x\to\pi/3}\left(\dfrac{\cos(x)-1/2}{x-\pi/3}\right)$. Use any method.

   Answer: $-\sqrt{3}/2$

   Solution: This limit represents the derivative computed at $x=\pi/3$ of the function $f(x)=\cos x$. Since the derivative of $f(x)$ is $-\sin x$, then its value at $x=\pi/3$ is exactly $-\sqrt{3}/2$.

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

   (a) Find the equation of the line tangent to the graph of $y=\tan(x)$ at $x=\dfrac{\pi}{4}$.

   Solution: We compute the derivative of $\tan(x)$ as being $\sec^2(x)$, which evaluated at $x=\frac{\pi}{4}$ yields 2. Since we also compute $\tan(\pi/4)=1$, then the equation of the tangent line is

   $$y-1=2\cdot(x-\pi/4).$$

   Marking scheme:

   - 1 mark for computing correctly the slope of the tangent.
   - 1 mark for correct equation of the tangent line.
   - Students do not need to simplify line equation.

   (b) For what values of x does the derivative of $\dfrac{x^2+6x+5}{\sin(x)}$ exist? Explain your answer.

Solution: The function is differentiable whenever $\sin(x) \neq 0$ since the derivative equals

$$\frac{\sin(x) \cdot (2x+6) - \cos(x) \cdot (x^2+6x+5)}{(\sin x)^2},$$

which is well-defined unless $\sin x = 0$. This happens when x is an integer multiple of π. So, the function is differentiable for all real values x except $x = n\pi, n \in \mathbb{Z}$.
Marking scheme:

- 1 mark for writing that the function is differentiable whenever the denominator is nonzero.
- 1 mark for solving correctly and finding **all** points where the function is not differentiable: $x = 0, \pm 1\pi, \pm 2\pi, \ldots$. Note that students can either write where derivative *exists* (eg writing $\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$) or where it *does not exist* (eg writing $n\pi$ for integer n).

Long answer question — you must show your work

3. [4 marks] Determine whether the derivative of the following function exists at $x = 0$

$$f(x) = \begin{cases} x^3 - 7x^2 & \text{if } x \leq 0 \\ x^3 \cos\left(\frac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

You must justify your answer using the definition of a derivative.

Solution: The function is differentiable at $x = 0$ if the following limit:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x) - 0}{x} = \lim_{x\to 0} \frac{f(x)}{x}$$

exists (note that we used the fact that $f(0) = 0$ as per the definition of the first branch which includes the point $x = 0$). We compute left and right limits; so

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{x^3 - 7x^2}{x} = \lim_{x\to 0^-} x^2 - 7x = 0$$

and

$$\lim_{x\to 0^+} \frac{x^3 \cos\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0^+} x^2 \cdot \cos\left(\frac{1}{x}\right).$$

This last limit equals 0 by the Squeeze Theorem since

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$$

and so,

$$-x^2 \leq x^2 \cdot \cos\left(\frac{1}{x}\right) \leq x^2,$$

where in these inequalities we used the fact that $x \to 0^+$ yields positive values for x. Finally, since $\lim_{x\to 0^+} -x^2 = \lim_{x\to 0^+} x^2 = 0$, the Squeeze Theorem yields that also $\lim_{x\to 0^+} x^2 \cos\left(\frac{1}{x}\right) = 0$, as claimed.

Since the left and right limits match (they're both equal to 0), we conclude that indeed $f(x)$ is differentiable at $x = 0$ (and its derivative at $x = 0$ is actually equal to 0). **Marking scheme:**

- 1 mark for writing the condition for differentiability, i.e. that the limit $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x}$ must exist. Not necessary to realise $f(0) = 0$ to get this mark.
- 1 mark for computing the left limit correctly.
- 1 mark for computing the right limit correctly. They lose 1 mark if they don't explain the use of the Squeeze Theorem.
- 1 mark for the correct answer (i.e. both left and right limits match for $\lim_{x\to 0} f(x)/x$, hence the function is differentiable at $x = 0$).

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Very short answer questions

1. [2 marks] Each part is worth 1 marks. Please write your **simplified** answers in the boxes.
Marking scheme: 1 for each correct, 0 otherwise

(a) Compute the derivative of $\left(\dfrac{x^2+3}{5x+2}\right)$

Answer: $\frac{5x^2+4x-15}{(5x+2)^2}$

Solution: We use quotient rule:

$$\frac{2x\cdot(5x+2)-(x^2+3)\cdot 5}{(5x+2)^2}=\frac{5x^2+4x-15}{(5x+2)^2}$$

(b) Evaluate $\lim_{x\to 2}\left(\dfrac{x^{2015}-2^{2015}}{x-2}\right)$. Use any method.

Answer: $2015\cdot 2^{2014}$

Solution: This limit represents the derivative computed at $x=2$ of the function $f(x)=x^{2015}$. Since the derivative of $f(x)$ is $2015\cdot x^{2014}$, then its value at $x=2$ is exactly $2015\cdot 2^{2014}$.

Short answer questions — you must show your work

2. [4 marks] Each part is worth 2 marks.

(a) Find the equation of the line tangent to the graph of $y=\cos(x)$ at $x=\dfrac{\pi}{4}$.

Solution: We compute the derivative of $\cos(x)$ as being $-\sin(x)$, which evaluated at $x=\frac{\pi}{4}$ yields $-\frac{1}{\sqrt{2}}$. Since we also compute $\cos(\pi/4)=1/\sqrt{2}$, then the equation of the tangent line is

$$y-1/\sqrt{2}=-1/\sqrt{2}\cdot(x-\pi/4).$$

Marking scheme:

- 1 mark for computing correctly the slope of the tangent.
- 1 mark for correct equation of the tangent line.
- Students do not need to simplify line equation.

(b) For what values of x does the derivative of $\dfrac{\sin(x)}{x^2+6x+5}$ exist? Explain your answer.

Solution: The function is differentiable whenever $x^2+6x+5 \neq 0$ since the derivative equals

$$\frac{\cos(x) \cdot (x^2+6x+5) - \sin(x) \cdot (2x+6)}{(x^2+6x+5)^2},$$

which is well-defined unless $x^2+6x+5=0$. We solve $x^2+6x+5=(x+1)(x+5)=0$, and get $x=-1$ and $x=-5$. So, the function is differentiable for all real values x except for $x=-1$ and for $x=-5$. **Marking scheme:**

- 1 mark for writing that the function is differentiable whenever the denominator is nonzero.
- 1 mark for solving correctly and finding **both** points where the function is not differentiable: $x=-1$ and $x=-5$. They can write their answer either as $f(x)$ is diferentiable for $x \in \mathbb{R} \setminus \{-1,-5\}$ or for $x \in (-\infty,-5) \cup (-5,-1) \cup (-1,+\infty)$, or even writing that $f(x)$ is differentiable for all x except for $x=-5,-1$.

Long answer question — you must show your work

3. [4 marks] Determine whether the derivative of the following function exists at $x=0$

$$f(x) = \begin{cases} 2x^3 - x^2 & \text{if } x \leq 0 \\ x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

You must justify your answer using the definition of a derivative.

Solution: The function is differentiable at $x=0$ if the following limit:

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)-0}{x} = \lim_{x\to 0} \frac{f(x)}{x}$$

exists (note that we used the fact that $f(0)=0$ as per the definition of the first branch which includes the point $x=0$). We compute left and right limits; so

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{2x^3-x^2}{x} = \lim_{x\to 0^-} 2x^2 - x = 0$$

and

$$\lim_{x\to 0^+} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0^+} x \cdot \sin\left(\frac{1}{x}\right).$$

This last limit equals 0 by the Squeeze Theorem since

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

and so,

$$-x \leq x \cdot \sin\left(\frac{1}{x}\right) \leq x,$$

where in these inequalities we used the fact that $x \to 0^+$ yields positive values for x. Finally, since $\lim_{x\to 0^+} -x = \lim_{x\to 0^+} x = 0$, the Squeeze Theorem yields that also $\lim_{x\to 0^+} x \sin\left(\frac{1}{x}\right) = 0$, as claimed.

Since the left and right limits match (they're both equal to 0), we conclude that indeed $f(x)$ is differentiable at $x = 0$ (and its derivative at $x = 0$ is actually equal to 0). **Marking scheme:**

- 1 mark for writing the condition for differentiability, i.e. that the limit $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x}$ must exist. Not necessary to realise $f(0) = 0$ to get this mark.
- 1 mark for computing the left limit correctly.
- 1 mark for computing the right limit correctly. They lose 1 mark if they don't explain the use of the Squeeze Theorem.
- 1 mark for the correct answer (i.e. both left and right limits match for $\lim_{x\to 0} f(x)/x$, hence the function is differentiable at $x = 0$).

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your **simplified** answers in the boxes.
Marking scheme: 1 for each correct, 0 otherwise

(a) Compute the derivative of $\left(\frac{3x^2+5}{2-x}\right)$

Answer: $\frac{-3x^2+12x+5}{(2-x)^2}$

Solution: We use quotient rule:

$$\frac{(2-x)(6x)-(3x^2+5)(-1)}{(2-x)^2} = \frac{-3x^2+12x+5}{(x-2)^2}$$

(b) Evaluate $\lim_{y\to 0}\left(\frac{\sqrt{100180+y}-\sqrt{100180}}{y}\right)$. Use any method.

Answer: $\frac{1}{2\sqrt{100180}}$

Solution: This limit represents the derivative computed at $x = 100180$ of the function $f(x) = \sqrt{x}$. Since the derivative of $f(x)$ is $\frac{1}{2\sqrt{x}}$, then its value at $x = 100180$ is exactly $\frac{1}{2\sqrt{100180}}$.

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

(a) Find the equation of the line tangent to the graph of $y = x^3$ at $x = \frac{1}{2}$.

Solution: We compute the derivative of x^3 as being $3x^2$, which evaluated at $x = \frac{1}{2}$ yields $\frac{3}{4}$. Since we also compute $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$, then the equation of the tangent line is

$$y - \frac{1}{8} = \frac{3}{4}\cdot\left(x - \frac{1}{2}\right).$$

Marking scheme:

- 1 mark for computing correctly the slope of the tangent.
- 1 mark for correct equation of the tangent line.
- Students do not need to simplify line equation.

(b) For what values of x does the derivative of $e^x \cdot (\sqrt{x} + \sin x)$ exist? Explain your answer.

Solution: The function is differentiable whenever $x > 0$ since the derivative equals

$$\begin{aligned} & e^x \left(\frac{1}{2\sqrt{x}} + \cos x \right) + e^x(\sqrt{x} + \sin x) \\ =& e^x \left(\frac{1}{2\sqrt{x}} + \cos x + \sqrt{x} + \sin x \right) \end{aligned}$$

which is well-defined unless $x \leq 0$. Note the function is defined at $x = 0$, but not differentiable there. **Marking scheme:**

- 1 mark for writing that x needs to be nonnegative for the derivative (indeed, the function) to exist
- 1 mark for differentiating correctly and realizing that, in the derivative, x cannot be 0.

Long answer question — you must show your work

3. [4 marks] Determine whether the derivative of the following function exists at $x = 0$

$$f(x) = \begin{cases} x \cos x & \text{if } x \geq 0 \\ \sqrt{x^2 + x^4} & \text{if } x < 0 \end{cases}$$

You must justify your answer using the definition of a derivative.

Solution: The function is differentiable at $x = 0$ if the following limit:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x) - 0}{x} = \lim_{x \to 0} \frac{f(x)}{x}$$

exists (note that we used the fact that $f(0) = 0$ as per the definition of the first branch which includes the point $x = 0$). We start by computing the left limit.

$$\lim_{x \to 0^-} \frac{f(x)}{x} = \lim_{x \to 0^-} \frac{\sqrt{x^2 + x^4}}{x} = \lim_{x \to 0^-} \frac{\sqrt{x^2}\sqrt{1 + x^2}}{x} = \lim_{x \to 0} \frac{-x\sqrt{1 + x^2}}{x} = -1$$

Now, from the right:

$$\lim_{x \to 0^+} \frac{x \cos x}{x} = \lim_{x \to 0^+} \cos x = 1.$$

Since the limit from the left does not equal the limit from the right, the derivative does not exist at $x = 0$. **Marking scheme:**

- 1 mark for writing the condition for differentiability, i.e. that the limit $\lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} \frac{f(x)}{x}$ must exist. Not necessary to realise $f(0) = 0$ to get this mark.
- 1 mark for computing the left limit correctly. They lose 1 mark if they get the sign wrong

- 1 mark for computing the right limit correctly.
- 1 mark for the correct answer (i.e. left and right limits disagree, hence the function is differentiable at $x = 0$).

First Name: ____________ Last Name: ____________

Student-No: ____________ Section: ____________

Very short answer questions

1. [2 marks] Each part is worth 1 marks. Please write your **simplified** answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

   (a) Compute the derivative of $\left(\dfrac{2-x^2}{3x^2+5}\right)$

   Answer: $\dfrac{-22x}{(3x^2+5)^2}$

   Solution: We use quotient rule:

   $$\frac{(3x^2+5)(-2x)-(2-x^2)(6x)}{(3x^2+5)^2} = \frac{-22x}{(3x^2+5)^2}$$

   (b) Evaluate $\lim_{t\to 4}\left(\dfrac{\frac{1}{\sqrt{t}}-\frac{1}{2}}{t-4}\right)$. Use any method.

   Answer: $-1/16$

   Solution: This limit represents the derivative computed at $t=4$ of the function $f(x)=1/\sqrt{t}$. Since the derivative of $f(t)$ is $-\frac{1}{2t^{3/2}}$, then its value at $t=4$ is exactly $-\frac{1}{2\cdot 4^{3/2}}=-1/16$.

Short answer questions — you must show your work

2. [4 marks] Each part is worth 2 marks.

   (a) Find the equation of the line tangent to the graph of $y=\sin(x)+\cos(x)+e^x$ at $x=0$.

   Solution: We compute the derivative $y'=\cos(x)-\sin(x)+e^x$, which evaluated at $x=0$ yields $1-0+1=2$. Since we also compute $y(0)=0+1+1=2$, the equation of the tangent line is

   $$y-2=2(x-0)$$

   ie $y=2x+2$. **Marking scheme:**

   - 1 mark for computing correctly the slope of the tangent.
   - 1 mark for correct equation of the tangent line.
   - Students do not need to simplify line equation.

(b) For what values of x does the derivative of $\frac{\sqrt{x}}{1-x^2}$ exist? Explain your answer.

Solution: The derivative of the function is

$$\frac{(1-x)^2/2\sqrt{x} - \sqrt{x}\cdot(-2x)}{(1-x^2)^2} = \frac{(1-x)^2 - 2x\cdot(-2x)}{2\sqrt{x}(1-x^2)^2}$$

The derivative is undefined if either $x < 0$ or $x = 0, \pm 1$ (since the square-root is undefined for $x < 0$ and the denominator is zero when $x = 0, 1, -1$. Putting this together — the derivative exists for $x > 0, x \neq 1$.

Marking scheme:

- 1 mark for writing that x needs to be nonnegative for the derivative (indeed, the function) to exist
- 1 mark for differentiating correctly and realizing that, in the derivative, x cannot be $0, 1$ (don't worry if they do not mention $x = -1$).

Long answer question — you must show your work

3. [4 marks] Determine whether the derivative of the following function exists at $x = 0$

$$f(x) = \begin{cases} x\cos x & \text{if } x \leq 0 \\ \sqrt{1+x} - 1 & \text{if } x > 0 \end{cases}$$

You must justify your answer using the definition of a derivative.

Solution: The function is differentiable at $x = 0$ if the following limit:

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{f(x)-0}{x} = \lim_{x\to 0}\frac{f(x)}{x}$$

exists (note that we used the fact that $f(0) = 0$ as per the definition of the first branch which includes the point $x = 0$).

We start by computing the left limit.

$$\lim_{x\to 0^-}\frac{f(x)}{x} = \lim_{x\to 0^-}\frac{x\cos x}{x} = \lim_{x\to 0^-}\cos x = 1.$$

Now, from the right:

$$\begin{aligned}\lim_{x\to 0^+}\frac{\sqrt{1+x}-1}{x} &= \lim_{x\to 0^+}\frac{\sqrt{1+x}-1}{x}\cdot\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} \\ &= \lim_{x\to 0^+}\frac{1+x-1}{x(\sqrt{1+x}+1)} = \lim_{x\to 0^+}\frac{1}{\sqrt{1+x}+1} = \frac{1}{2}\end{aligned}$$

Since the limit from the left does not equal the limit from the right, the derivative does not exist at $x = 0$. **Marking scheme:**

- 1 mark for writing the condition for differentiability, i.e. that the limit $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x}$ must exist. Not necessary to realise $f(0) = 0$ to get this mark.
- 1 mark for computing the left limit correctly.
- 1 mark for computing the right limit correctly.
- 1 mark for the correct answer (left and right limits don't match so not differentiable).