Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

   (a) Compute
   $$\lim_{x \to +\infty} \frac{x^3 + 2x^2 - 1}{4x^3 + 3x + 5}.$$

   Answer: $\frac{1}{4}$

   Solution: We have, after dividing both numerator and denominator by x^3 (which is the highest power of the denominator) that

   $$\frac{x^3 + 2x^2 - 1}{4x^3 + 3x + 5} = \frac{1 + \frac{2}{x} - \frac{1}{x^3}}{4 + \frac{3}{x^2} + \frac{5}{x^3}}.$$

   Since $1/x^n \to 0$ as $x \to +\infty$, we conclude that

   $$\lim_{x \to +\infty} \frac{x^3 + 2x^2 - 1}{4x^3 + 3x + 5} = \frac{1}{4}.$$

   (b) Compute the derivative of $\left(\dfrac{x - 2}{3x^2 + x} \right)$

   Answer: $\frac{-3x^2 + 12x + 2}{(3x^2 + x)^2}$

   Solution: We use the quotient rule:

   $$\frac{1 \cdot (3x^2 + x) - (x - 2)(6x + 1)}{(3x^2 + x)^2} = \frac{-3x^2 + 12x + 2}{(3x^2 + x)^2}.$$

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

   (a) Evaluate
   $$\lim_{x \to -\infty} \frac{8x - 5}{\sqrt{4x^2 + x} - 6}.$$

   Solution: We divide by the highest power of the denominator, which is x and note that (for $x < 0$)
   $$\frac{\sqrt{4x^2 + x}}{x} = -\sqrt{\frac{4x^2 + x}{x^2}} = -\sqrt{4 + \frac{1}{x}}.$$
   Since $1/x \to 0$ as $x \to -\infty$, we conclude that
   $$\lim_{x \to -\infty} \frac{8x - 5}{\sqrt{4x^2 + x} - 6} = \lim_{x \to -\infty} \frac{8 - \frac{5}{x}}{-\sqrt{4 + \frac{1}{x}} - \frac{6}{x}} = \frac{8}{-2} = -4.$$

(b) Find the equation of the tangent line to the graph of $y = x^3 - 2x^2 - 1$ at $x = 2$.

Solution: We compute the derivative of $x^3 - 2x^2 - 1$ as being $3x^2 - 4x$, which evaluated at $x = 2$ yields 4. Since we also compute $2^3 - 2 \cdot 2^2 - 1 = -1$, then the equation of the tangent line is

$$y + 1 = 4(x - 2) \qquad \text{or} \qquad y = 4x - 9.$$

Long answer question — you must show your work

3. ⬛ 4 marks ⬛ Show that there exists at least one real number c such that $2\cos(\frac{c}{2}) = \sin(c) - \frac{1}{c}$.

Solution: We let $f(x) = 2\cos(\frac{x}{2}) - \sin(x) + \frac{1}{x}$. Then $f(x)$ is continuous on the positive real numbers since $\cos(x)$ and $\sin(x)$ are everywhere continuous and $\frac{1}{x}$ is continuous everywhere except at $x = 0$.

We find a positive value a such that $f(a) > 0$. We observe that $a = \pi$ works since

$$f(\pi) = 2\cos(\frac{\pi}{2}) - \sin(\pi) + \frac{1}{\pi} = 0 + 0 + \frac{1}{\pi} > 0.$$

We find a positive value b such that $f(b) < 0$. We see that $b = 2\pi$ works since

$$f(2\pi) = 2\cos(\pi) - \sin(2\pi) + \frac{1}{2\pi} = -2 + 0 + \frac{1}{2\pi} < 0,$$

because $\frac{1}{2\pi} < 2$.

So, because $f(x)$ is continuous on $[\pi, 2\pi]$ and $f(\pi) < 0$ while $f(2\pi) > 0$, then the Intermediate Value Theorem guarantees the existence of a real number $c \in (\pi, 2\pi)$ such that $f(c) = 0$. **Marking scheme:**

- 1 mark for finding a value $a \in I$ such that $f(a) > 0$.

- 1 mark for finding a value $b \in I$ such that $f(b) < 0$.

- 1 mark for the correct conclusion which **should** mention that the solution c is in between a and b **and** its existence is justified by the Intermediate Value Theorem.

- If a and b are on different sides of 0, give at most 2 points total.