MATH 532 – HOMEWORK SET 3

FEBRUARY 24, 2016

1) (a) Let $f: X \to S$ and $g: Y \to S$ be morphisms of quasi-projective algebraic sets. Describe (without proof) in more common terms the fiber product $X \times_S Y$ in the following cases:

   (A) $S = \mathbb{A}^0 = \mathrm{pt}$

   (B) $X = \mathbb{A}^0 = \mathrm{pt}$

   (C) f is an immersion.

   (D) f and g are immersions.

   (b) Let $X \to T \to S$ and $Y \to S$ be morphisms. Prove that

$$X \times_T (T \times_S Y) \cong X \times_S Y.$$

   (c) Let $f: X \to Y$ and $g: Y \to Z$ be proper morphisms. Prove that $g \circ f$ is proper.

2) Let $\mathrm{Grass}(2,4) \subset \mathbb{P}(\bigwedge^2 k^4) = \mathbb{P}^5$ be the Grassmannian of 2-planes in k^4. Let e_i be the standard basis of k^4 and $e_i \wedge e_j$ $(1 \leq i < j \leq 4)$ be the induced basis of $\bigwedge^2 k^4$. Finally write $x_{i,j}$ for the corresponding homogeneous coordinates of $\mathbb{P}(\bigwedge^2 k^4) = \mathbb{P}^5$ (called *Plücker coordinates*).

   Show that $\mathrm{Grass}(2,4) = \widetilde{Z}(x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{1,4}x_{2,3}) \subseteq \mathbb{P}^5$.

   (It might be useful to read Chapter 8 of [G] to gain more familiarity with the exterior product.)

3) Locate the singular points of the following surfaces in $\mathbb{A}^3$ (assume $\mathrm{char}\, k \neq 2$). Which is which in Figure 1?

   (a) $xy^2 = z^2$;

   (b) $x^2 + y^2 = z^2$;

   (c) $xy + x^3 + y^3 = 0$.

4) Let P be a point of an irreducible algebraic set X and let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_{X,P}$. The *Zariski tangent space* $T_P(X)$ of X at P is the dual k-vector space of $\mathfrak{m}/\mathfrak{m}^2$.

   (a) Show that $\dim T_P(X) \geq \dim X$ with equality if and only if P is non-singular.

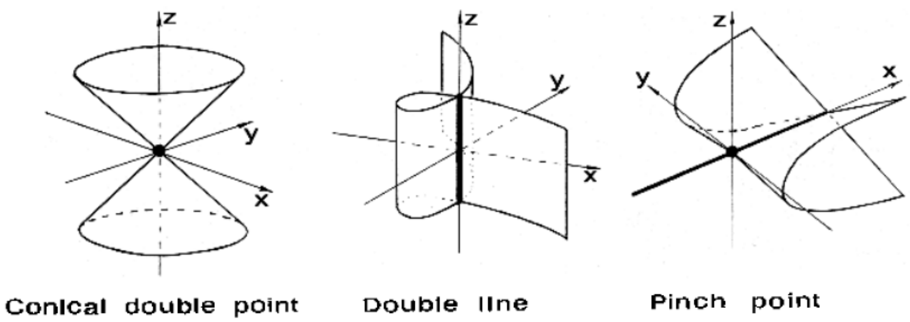

Figure 1: Surface singularities (cf. [H, p. 36])

(b) Show that for any morphism $f: X \to Y$ there is a natural induced k-linear map $T_P(f): T_P(X) \to T_{f(P)}(Y)$.

(c) If f is the vertical projection of the parabola $x = y^2$ onto the x-axis, show that the induced map $T_0(f)$ of tangent spaces at the origin is the zero map.

5) (a) Show that if $f: \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ is a rational map, then there exists a unique extension of f to a morphism $\mathbb{P}^1 \to \mathbb{P}^1$.

(b) Think of $\mathbb{P}^1$ as $\mathbb{A}^1 \cup \{\infty\}$. Then we define a *fractional linear transformation* of $\mathbb{P}^1$ by sending $x \mapsto \frac{ax+b}{cx+d}$ for $a, b, c, d \in k$ with $ad - bc \neq 0$. Show that a fractional linear transformation induces an *automorphism* of $\mathbb{P}^1$. Further show that the fractional linear transformations form a group isomorphic to $\mathrm{PGL}_2(k) = \mathrm{GL}_2(k)/k^*$.

(c) Let $\mathrm{Aut}\,\mathbb{P}^1$ denote the group of all automorphisms of $\mathbb{P}^1$. Show that $\mathrm{Aut}\,\mathbb{P}^1 \cong \mathrm{Aut}\,k(x)$, the group of all k-algebra automorphisms of the field $k(x)$.

(d) Now show that every automorphism of $k(x)$ is a fractional linear transformation, and deduce that $\mathrm{PGL}_2(k) \to \mathrm{Aut}\,\mathbb{P}^1$ is an isomorphism.

REFERENCES

[G] Andreas Gathmann. *Algebraic Geometry. Class Notes TU Kaiserslautern 2014.* URL: http://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2014/main.pdf.

[H] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics 52. New York: Springer-Verlag, 1977. xvi+496. ISBN: 0-387-90244-9.